Errors in Hypothesis Testing: Type I and Type II Errors

When conducting a hypothesis test, there are two acceptable conclusions to come to:

1. There is sufficient evidence to reject the null hypothesis at the specified significance level.
2. There is insufficient evidence to reject the null hypothesis at the specified significance level.

However, it is possible for these conclusions to be wrong. It isn't possible to eliminate these errors, however it is possible to calculate the likelihood of them existing.

		Reality	
		H_0 is correct	H_0 is incorrect
Conclusion from hypothesis test	1. Reject H_0	**Type I Error**	Correct conclusion
	2. Insufficient evidence to reject H_0	Correct conclusion	**Type II Error**

Type I (one) errors are when H_0 is rejected, when it is in fact correct.

Type II (two) errors are when H_0 is not rejected, however it should have been.

This cheat sheet will be focusing on Type I errors.

Probability of Type I Errors

A Type I error occurs when the test statistic falls within the rejection region when H_0 is in fact true. The critical region in a hypothesis test corresponds to the significance level that the hypothesis test was conducted to, therefore the significance level is the probability of a Type I error. However, it must be noted that when dealing with discrete random variables, the significance level stated in the hypothesis test is not equal to the actual significance level of the test. The significance level will be the largest significance level possible that is less than the declared significance level. Therefore, the probability of a Type I error is given by:

$$\text{Actual significance level} = \text{P(type I error)} = \text{P(rejecting } H_0 | H_0 \text{ is true)}$$

Additionally, the significance level is not always given in a question, so it would need to be calculated in this way.

Example 1: Given that $X \sim N(\mu, 16)$ find the probability of a type I error for the following hypothesis test:

$$H_0: \mu = 8; \text{ reject } H_0 \text{ if } X > 14$$

Use the definition of a type I error and evaluate the terms.	$\text{P(type I error)} = \text{P(rejecting } H_0 \mid H_0 \text{ is true)}$ $= \text{P}(X > 14)$ given that $X \sim N(8, 16)$
Use a calculator or datasheet.	$= 0.066807$ $= 0.0668$ to 3d. p.

Example 2: The number of calls arriving at a company call centre in an hour follows a Poisson distribution with mean 15. The company has made an online FAQ section and the management wants to test whether the number of calls per hour has decreased.

a) State the appropriate null and alternative hypotheses for this test.
b) The management decide to note the number of calls in a random hour of the day and reject the null hypothesis if the number of calls is 9 or less. Find the significance level of the hypothesis test, and hence the probability of a type I error.

a) The significance level of the test has not been given, so the hypotheses need to be stated in terms of the mean.	$H_0: \lambda = 15;\ H_1: \lambda < 15$
b) Calculate the probability that the number of observed calls is less than or equal to 9.	If X is the number of calls received in an hour, and if H_0 is true $X \sim Po(15)$. For this distribution $P(X \leq 9) = 0.0698$ Therefore, the significance level of the test is 6.98%.
The probability of type I error is given by:	$\text{P(type I error)} = \text{P(rejecting } H_0 \mid H_0 \text{ is true)} = 0.0698$

Example 3: A hypothesis test is conducted in order to see whether a six-sided die is biased towards rolling a "one". The dice is rolled 24 times and the test is conducted to a 5% significance level.

a) State the appropriate null and alternative hypotheses for this test.
b) How many ones would need to be seen to reject the null hypothesis?
c) Find the probability of type I error

a) The test is to see whether the probability is higher than it should be, so this is a one-tailed hypothesis test.	$H_0: p = \frac{1}{6};\ H_1: p > \frac{1}{6}$
b) Due to the discrete nature of the binomial distribution, it is not possible to test at exactly a 5% significance level. Because of this, it is required to use the given distribution to find the value of X that gives the largest possible significance level that is less than 5%.	If X is the number of ones that are rolled, and if H_0 is true: $X \sim B(24, \frac{1}{6})$
Use a calculator to calculate $P(X \geq x)$ for different values of x.	x / $P(X \geq x)$: 5 / 0.1995; 6 / 0.0912; 7 / 0.0354; 8 / 0.0118 This shows that the number of ones that needs to be rolled in order to reject the null hypothesis is 7.
c) The probability of type I error is equal to the actual significance level of the hypothesis test.	$\text{P(type I error)} = \text{P(rejecting } H_0 \mid H_0 \text{ is true)} = 0.0354 = 3.54\%$

x	$P(X \geq x)$
5	0.1995
6	0.0912
7	0.0354
8	0.0118

Example 4: A hypothesis test is conducted in order to see whether a new set of traffic lights has led to less traffic accidents on one particular road. Before the lights were installed, there was an average rate of 6 per month. This month, 4 accidents occurred

a) State the appropriate null and alternative hypotheses for this test.
b) Perform a hypothesis test to see whether or not less accidents occurred, to the 5% significance level.
c) Find the probability of type I error for this test.

a) The test is to see whether the rate of accidents has decreased, following a Poisson distribution.	$H_0: \lambda = 6;\ H_1: \lambda < 6$
b) Calculate the probability that $X \leq 4$, and compare to 0.05.	$X \sim Po(6)$ $P(X \leq 4) = 0.285$ $0.285 > 0.05$ therefore, do not reject the null hypothesis.
c) Use a calculator to calculate $P(X \leq x)$ for different values of x, in order to find out the first value of X that falls within the critical region. The probability of type I error is equal to the actual significance level of the hypothesis test.	The first value that falls below the 5% cut-off is for when $X \leq 1$. This means that $\text{P(type I error)} = 0.0174$

x	$P(X \leq x)$
3	0.1512
2	0.6197
1	0.0174